5 Implicit Differentiation

Learning Objectives: After completing this section, we should be able to

- find the derivative of a function when it is implicitly defined.

5.1 Chain Rule Review

Chain Rule Problem from a Table:

Example. We have $f'(1) = 2$, $g(0) = 1$, and $g'(0) = 3$. Compute $\frac{d}{dx}(f(x+g(x)))$ at $x = 0$.

inside = $x+g(x)$ outside = $f(u)$

inside′ = $1+g'(x)$ outside′ = $f'(u)$

By the chain rule,

$$\frac{d}{dx} f(x+g(x)) = f'(x+g(x)) \cdot (1+g'(x))$$

outside′(inside) · inside′

$$\text{So } \frac{d}{dx} f(x+g(x))\Big|_{\underbrace{x=0}_{\text{"evaluate at"}}} = f'(0+g(0)) \cdot (1+g'(0))$$

$$= f'(0+1) \cdot (1+3) \quad \left(\text{using } \begin{array}{l} g(0)=1 \\ g'(0)=3 \end{array}\right)$$

$$= f'(1) \cdot 4$$

$$= 2 \cdot 4 = 8 \quad (\text{using } f'(1)=2)$$

You try!

Example. Let $f(x) = x^2 + 2(g(x))^3$. Find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[x^2 + 2\cdot(g(x))^3\right]$$

$$= \frac{d}{dx}x^2 + \frac{d}{dx}2\cdot(g(x))^3$$

$$= 2x + \text{chain rule!}$$

chain rule

$\frac{d}{dx} f(g(x))$
$= f'(g(x)) \cdot g'(x)$

so $\frac{d}{dx}$ outside(inside)
= outside′(inside) · inside′

outside = $2 \cdot u^3$ inside = $g(x)$

outside′ = $6u^2$ inside′ = $g'(x)$

$$\Rightarrow \frac{d}{dx} 2(g(x))^3 = \text{outside}'(\text{inside}) \cdot \text{inside}'$$

$$= \text{outside}'(g(x)) \cdot g'(x)$$

$$= 6(g(x))^2 \cdot g'(x)$$

So $f'(x) = 2x + 6(g(x))^2 \cdot g'(x)$.

5.2 Implicit Equations

An **explicit equation** is of the form $y =$ some function of x

Ex) $y = e^{\cos^2(x^4)}$

dependent variable

function that only depends on the the independent variable x.

An **implicit equation** is a relation/equation involving y & x, but it is not written in the form $y =$ some fnctn of x

Ex) $x^2 + y^2 = 1$ (a circle of radius 1)

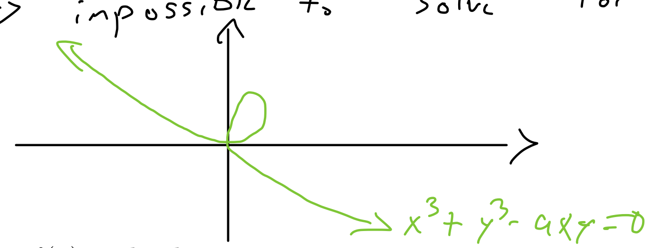
What happens if we try to solve for y?

$y^2 = 1 - x^2$

$\Rightarrow y = \pm\sqrt{1-x^2} \Rightarrow y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$

2 different equations!

Ex) $x^3 + y^3 - 9xy = 0 \Rightarrow$ impossible to solve for y



Even though we don't know $y = f(x)$ explicitly, we can still find the slope of the tangent line to an implicit curve if we find $\frac{dy}{dx}$.

Example. $x^3 + y^3 - 9xy = 0$. Find $\frac{dy}{dx}$.

Still think of y as depending on x, even though we can't write it as $y =$ some fnctn of x

$\Rightarrow$ Think of y as if it were inside of a chain, i.e., y^3 is some "fnctn" y inside an outer fnctn

u^3

Example Continued.

Step 1) Take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}\left(x^3 + y^3 - 9xy\right) = \frac{d}{dx}(0)$$

$$\Rightarrow \underbrace{\frac{d}{dx} x^3}_{\Downarrow} + \underbrace{\frac{d}{dx} y^3}_{\Downarrow} - 9 \underbrace{\frac{d}{dx} x \cdot y}_{\text{product rule}} = 0$$

straight forward

$3x^2$

chain

outside $= f(u) = u^3$

outside$' = f'(u) = 3u^2$

inside $= y$

inside$' = y' = \frac{d}{dx} y = \frac{dy}{dx}$

$$\frac{d}{dx} y^3 = \text{outside}'(\text{inside}) \cdot \text{inside}'$$

$$= \text{outside}'(y) \cdot \frac{dy}{dx}$$

$$= 3y^2 \frac{dy}{dx}$$

$\frac{d}{dx}(x \cdot y) =$ product rule

left $= x$ right $= y$

left$' = 1$ right$' = y' = \frac{dy}{dx}$

$$\frac{d}{dx}(x \cdot y) = \text{left} \cdot \text{right}' + \text{left}' \cdot \text{right}$$

$$= x \cdot \frac{dy}{dx} + 1 \cdot y$$

$$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} \cancel{+} - 9\left(x \cdot \frac{dy}{dx} + 1 \cdot y\right) = 0$$

$$\Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} - 9x \cdot \frac{dy}{dx} - 9 \cdot 1 \cdot y = 0$$

Step 2: Solve for $\frac{dy}{dx}$

(a) Get all terms involving $\frac{dy}{dx}$ to one side of the equation, and every other term to the other side

(b) factor $\frac{dy}{dx}$

(c) divide to solve for $\frac{dy}{dx}$

$$3x^2 + 3y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} - 9y = 0$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} = -3x^2 + 9y \qquad (a)$$

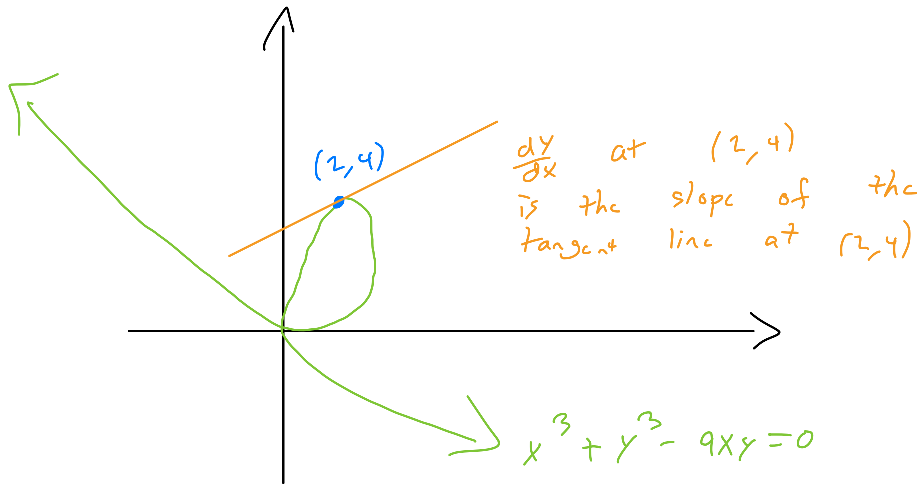
$$\Rightarrow \frac{\frac{dy}{dx}\left[3y^2 - 9x\right]}{3y^2 - 9x} = \frac{-3x^2 + 9y}{3y^2 - 9x} \qquad (b)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-3x^2 + 9y}{3y^2 - 9x} \qquad (c)$$

Example Continued. What does this $\frac{dy}{dx}$ tell us?

The formula $\frac{dy}{dx}$ gives slope of the tangent to the implicit curve, so long as (x, y) is on the original curve

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$



To find $\frac{dy}{dx}$ at a point, always verify the point is on the curve:

Ex) Verify (2,4) is on $x^3 + y^3 - 9xy = 0$

$$\Rightarrow 2^3 + 4^3 - 9 \cdot 2 \cdot 4 = 8 + 64 - 72 = 0 \checkmark$$

Find $\frac{dy}{dx}$ at (2,4):

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{9y - 3x^2}{3y^2 - 9x}\right|_{(2,4)} = \frac{9 \cdot 4 - 3 \cdot 2^2}{3 \cdot 4^2 - 9 \cdot 2} = \frac{24}{30} = \frac{4}{5}$$

"evaluated at (2,4)"

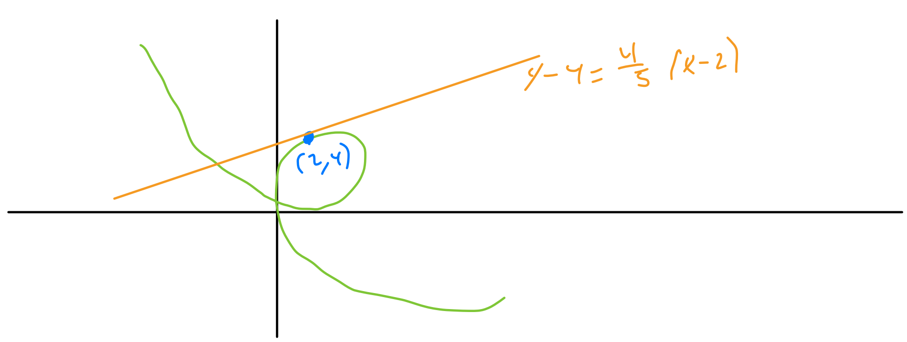
What's the tangent line to the implicit curve at (2,4)?

point: (2, 4)

slope: 4/5

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{4}{5}(x - 2)$$



Example. Suppose $y^2e^x + xe^{y^2} = 4 + x$. Find $\frac{dy}{dx}$.

Step 1

$$\frac{d}{dx}\left(y^2e^x + xe^{y^2}\right) = \frac{d}{dx}(4+x)$$

$$\Rightarrow \underbrace{\frac{d}{dx} y^2e^x}_{\text{product rule}} + \underbrace{\frac{d}{dx} x e^{y^2}}_{\text{product rule}} = 0 + 1$$

product rule

left $= y^2$ right $= e^x$

left$' = 2y\frac{dy}{dx}$ right$' = e^x$

$y^2e^x + 2y\frac{dy}{dx}\cdot e^x$

product rule

left $= x$ right $= e^{y^2}$

left$' = 1$ right$' = \underbrace{e^{y^2}}_{\text{original exponent}} \cdot \underbrace{\left(2y\frac{dy}{dx}\right)}_{\text{derivative of exponent}}$

$\Rightarrow x\cdot e^{y^2}\cdot 2y\frac{dy}{dx} + 1\cdot e^{y^2}$

$$\Rightarrow y^2e^x + 2y\frac{dy}{dx}e^x + x\left(e^{y^2}\cdot 2y\frac{dy}{dx}\right) + e^{y^2} = 1$$

2a) $\Rightarrow 2y\frac{dy}{dx}e^x + xe^{y^2}\cdot 2y\frac{dy}{dx} = 1 - y^2e^x - e^{y^2}$

2b) $\Rightarrow \frac{dy}{dx}\left[2ye^x + xe^{y^2}\cdot 2y\right] = 1 - y^2e^x - e^{y^2}$

2c) $\frac{dy}{dx} = \frac{1 - y^2e^x - e^{y^2}}{2ye^x + xe^{y^2}\cdot 2y}$

You try!

Example. Suppose $\tan(x^2y) = x + y^3\sin(x)$. Find $\frac{dy}{dx}$.

Example. The energy of a capacitor is given by $E = \frac{1}{2}\frac{Q^2}{C}$.